

# Instability of solitary waves for nonlinear Schrödinger equations of derivative type

Dedicated to Professor Nakao Hayashi on his sixtieth birthday

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## Abstract

We study the orbital stability and instability of solitary wave solutions for nonlinear Schrödinger equations of derivative type.

## 1 Introduction

In this paper, we study the instability of solitary wave solutions for nonlinear Schrödinger equations of the form

$$i\partial_t u = -\partial_x^2 u - i|u|^2 \partial_x u - b|u|^4 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.1)$$

where  $b \geq 0$  is a constant. Eq. (1.1) appears in various areas of physics such as plasma physics, nonlinear optics, and so on (see, e.g., [12, 13] and also Introduction of [16]). It is known that (1.1) has a two parameter family of solitary wave solutions

$$u_\omega(t, x) = e^{i\omega_0 t} \phi_\omega(x - \omega_1 t), \quad (1.2)$$

where  $\omega = (\omega_0, \omega_1) \in \Omega := \{(\omega_0, \omega_1) \in \mathbb{R}^2 : \omega_1^2 < 4\omega_0\}$ ,  $\gamma = 1 + \frac{16}{3}b$ ,

$$\phi_\omega(x) = \tilde{\phi}_\omega(x) \exp\left(i\frac{\omega_1}{2}x - \frac{i}{4} \int_{-\infty}^x |\tilde{\phi}_\omega(\eta)|^2 d\eta\right), \quad (1.3)$$

$$\tilde{\phi}_\omega(x) = \left\{ \frac{2(4\omega_0 - \omega_1^2)}{-\omega_1 + \sqrt{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)} \cosh(\sqrt{4\omega_0 - \omega_1^2} x)} \right\}^{1/2}. \quad (1.4)$$

Here, we note that  $\phi_\omega(x)$  is a solution of

$$-\partial_x^2 \phi + \omega_0 \phi + \omega_1 i \partial_x \phi - i|\phi|^2 \partial_x \phi - b|\phi|^4 \phi = 0, \quad x \in \mathbb{R}, \quad (1.5)$$

and  $\tilde{\phi}_\omega(x)$  is a solution of

$$-\partial_x^2 \phi + \frac{4\omega_0 - \omega_1^2}{4} \phi + \frac{\omega_1}{2} |\phi|^2 \phi - \frac{3}{16} \gamma |\phi|^4 \phi = 0, \quad x \in \mathbb{R}. \quad (1.6)$$

For  $v, w \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$ , we define

$$(v, w)_{L^2} = \Re \int_{\mathbb{R}} v(x) \overline{w(x)} dx,$$

and regard  $L^2(\mathbb{R})$  as a real Hilbert space. Similarly,  $H^1(\mathbb{R}) = H^1(\mathbb{R}, \mathbb{C})$  is regarded as a real Hilbert space with inner product

$$(v, w)_{H^1} = (v, w)_{L^2} + (\partial_x v, \partial_x w)_{L^2}.$$

We define the energy  $E : H^1(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$E(v) = \frac{1}{2} \|\partial_x v\|_{L^2}^2 - \frac{1}{4} (i|v|^2 \partial_x v, v)_{L^2} - \frac{b}{6} \|v\|_{L^6}^6. \quad (1.7)$$

Then, we have

$$E'(v) = -\partial_x^2 v - i|v|^2 \partial_x v - b|v|^4 v,$$

and (1.1) can be written in a Hamiltonian form  $i\partial_t u = E'(u)$  in  $H^{-1}(\mathbb{R})$ .

For  $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$  and  $v \in H^1(\mathbb{R})$ , we define

$$T(\theta)v(x) = e^{i\theta_0} v(x - \theta_1) \quad (x \in \mathbb{R}). \quad (1.8)$$

Note that the energy  $E$  is invariant under  $T$ , i.e.,

$$E(T(\theta)v) = E(v), \quad \theta \in \mathbb{R}^2, v \in H^1(\mathbb{R}), \quad (1.9)$$

and that the solitary wave solution (1.2) is written as  $u_\omega(t) = T(\omega t)\phi_\omega$ .

The Cauchy problem for (1.1) is locally well-posed in the energy space  $H^1(\mathbb{R})$  (see [16] and also [7, 8, 9]). For any  $u_0 \in H^1(\mathbb{R})$ , there exist  $T_{\max} \in (0, \infty]$  and a unique solution  $u \in C([0, T_{\max}), H^1(\mathbb{R}))$  of (1.1) with  $u(0) = u_0$  such that either  $T_{\max} = \infty$  or  $T_{\max} < \infty$  and  $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{H^1} = \infty$ . Moreover, the solution  $u(t)$  satisfies

$$E(u(t)) = E(u_0), \quad Q_0(u(t)) = Q_0(u_0), \quad Q_1(u(t)) = Q_1(u_0)$$

for all  $t \in [0, T_{\max})$ , where  $Q_0$  and  $Q_1$  are defined by

$$Q_0(v) = \frac{1}{2}\|v\|_{L^2}^2, \quad Q_1(v) = \frac{1}{2}(i\partial_x v, v)_{L^2}. \quad (1.10)$$

For  $\varepsilon > 0$ , we define

$$U_\varepsilon(\phi_\omega) = \{u \in H^1(\mathbb{R}) : \inf_{\theta \in \mathbb{R}^2} \|u - T(\theta)\phi_\omega\|_{H^1} < \varepsilon\}.$$

Then, the stability and instability of solitary waves are defined as follows.

**Definition 1.** We say that the solitary wave solution  $T(\omega t)\phi_\omega$  of (1.1) is *stable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $u_0 \in U_\delta(\phi_\omega)$ , then the solution  $u(t)$  of (1.1) with  $u(0) = u_0$  exists for all  $t \geq 0$ , and  $u(t) \in U_\varepsilon(\phi_\omega)$  for all  $t \geq 0$ . Otherwise,  $T(\omega t)\phi_\omega$  is said to be *unstable*.

For the case  $b = 0$ , Colin and Ohta [2] proved that the solitary wave solution  $T(\omega t)\phi_\omega$  of (1.1) is stable for all  $\omega \in \Omega$  (see also [6, 20]). We remark that the instability of solitary waves for (1.1) is not studied in previous papers [2, 6, 20]. For a recent result on a generalized derivative nonlinear Schrödinger equation, see [10].

In this paper, we consider the case  $b > 0$ , and prove the following.

**Theorem 1.** *Let  $b > 0$ . Then there exists  $\kappa = \kappa(b) \in (0, 1)$  such that the solitary wave solution  $T(\omega t)\phi_\omega$  of (1.1) is stable if  $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$ , and unstable if  $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$ .*

**Remark 1.** Let  $b > 0$ ,  $\gamma = 1 + \frac{16}{3}b$ , and

$$g(\xi) = \frac{2(\gamma - 1)}{\xi} \tan^{-1} \frac{1 + \sqrt{1 + \xi^2}}{\xi}, \quad \xi \in (0, \infty). \quad (1.11)$$

Then,  $g : (0, \infty) \rightarrow (0, \infty)$  is strictly decreasing and bijective. Thus, for any  $b > 0$ , there exists a unique  $\hat{\xi} = \hat{\xi}(b) \in (0, \infty)$  such that  $g(\hat{\xi}) = 1$ . The constant  $\kappa$  in Theorem 1 is given by  $\kappa = (1 + \hat{\xi}^2/\gamma)^{-1/2}$  (see Lemma 1 below).

**Remark 2.** The sufficient condition  $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$  for stability of  $T(\omega t)\phi_\omega$  is equivalent to  $Q_1(\phi_\omega) > 0$ , and the sufficient condition  $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$  for instability is equivalent to  $Q_1(\phi_\omega) < 0$  (see Lemma 1 and Proof of Theorem 1 below). We also remark that  $E(\phi_\omega) = -\frac{\omega_1}{2}Q_1(\phi_\omega)$  for all  $\omega \in \Omega$ .

**Remark 3.** We do not study the borderline case  $\omega_1 = 2\kappa\sqrt{\omega_0}$  in this paper, and leave it as an open problem. Note that  $E(\phi_\omega) = Q_1(\phi_\omega) = 0$  in the case  $\omega_1 = 2\kappa\sqrt{\omega_0}$ . For related results for one-parameter family of solitary waves in borderline cases, see [1, 15, 14, 11].

**Remark 4.** It is not known whether (1.1) has finite time blowup solutions or not. It will be interesting to study relations between unstable solitary wave solutions obtained in Theorem 1 and the existence of blowup solutions for (1.1). For a recent progress in this direction, see Wu [18, 19].

For  $\omega \in \Omega$ , we define the action  $S_\omega : H^1(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$S_\omega(v) = E(v) + \sum_{j=0}^1 \omega_j Q_j(v),$$

where  $E$ ,  $Q_0$  and  $Q_1$  are defined by (1.7) and (1.10). Note that  $Q'_0(v) = v$ ,  $Q'_1(v) = i\partial_x v$ , and that (1.5) is equivalent to  $S'_\omega(\phi) = 0$ .

We also define a function  $d : \Omega \rightarrow \mathbb{R}$  by

$$d(\omega) = S_\omega(\phi_\omega) = E(\phi_\omega) + \sum_{j=0}^1 \omega_j Q_j(\phi_\omega).$$

Then, we have

$$d'(\omega) = (\partial_{\omega_0} d(\omega), \partial_{\omega_1} d(\omega)) = (Q_0(\phi_\omega), Q_1(\phi_\omega)),$$

and the Hessian matrix  $d''(\omega)$  of  $d(\omega)$  is given by

$$d''(\omega) = \begin{bmatrix} \partial_{\omega_0}^2 d(\omega) & \partial_{\omega_1} \partial_{\omega_0} d(\omega) \\ \partial_{\omega_0} \partial_{\omega_1} d(\omega) & \partial_{\omega_1}^2 d(\omega) \end{bmatrix} = \begin{bmatrix} \partial_{\omega_0} Q_0(\phi_\omega) & \partial_{\omega_1} Q_0(\phi_\omega) \\ \partial_{\omega_0} Q_1(\phi_\omega) & \partial_{\omega_1} Q_1(\phi_\omega) \end{bmatrix}.$$

To prove Theorem 1, we use the following sufficient conditions for stability and instability in terms of the Hessian matrix  $d''(\omega)$  (see [5]).

**Theorem 2.** *Let  $\omega \in \Omega$ . If the matrix  $d''(\omega)$  has a positive eigenvalue, then the solitary wave solution  $T(\omega t)\phi_\omega$  of (1.1) is stable.*

**Theorem 3.** *Let  $\omega \in \Omega$ . If  $d''(\omega)$  is negative definite (all eigenvalues of  $d''(\omega)$  are negative), then the solitary wave solution  $T(\omega t)\phi_\omega$  of (1.1) is unstable.*

Theorem 2 can be proved in the same way as in Colin and Ohta [2], and we omit the proof. We give the proof of Theorem 3 in Section 3 below. As we stated above, the instability of solitary waves for (1.1) has not been studied in previous papers [2, 6, 20].

Moreover, by the explicit form (1.3) with (1.4) of  $\phi_\omega$ , and by elementary computations, we have the following.

**Lemma 1.** *Let  $b > 0$  and  $\gamma = 1 + \frac{16}{3}b$ . For  $\omega \in \Omega$ , we have*

$$\begin{aligned} Q_0(\phi_\omega) &= \frac{4}{\sqrt{\gamma}} \tan^{-1} \frac{\omega_1 + \sqrt{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)}}{\sqrt{\gamma(4\omega_0 - \omega_1^2)}}, \\ Q_1(\phi_\omega) &= \frac{1}{\gamma^{3/2}} \left\{ \sqrt{\gamma(4\omega_0 - \omega_1^2)} \right. \\ &\quad \left. - 2(\gamma - 1)\omega_1 \tan^{-1} \frac{\omega_1 + \sqrt{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)}}{\sqrt{\gamma(4\omega_0 - \omega_1^2)}} \right\}, \\ \det[d''(\omega)] &= \frac{-\gamma Q_1(\phi_\omega)}{\sqrt{\gamma(4\omega_0 - \omega_1^2)}\{\omega_1^2 + \gamma(4\omega_0 - \omega_1^2)\}}. \end{aligned}$$

Theorem 1 follows from Theorems 2 and 3, Lemma 1 and Remark 1.

*Proof of Theorem 1.* Let  $\omega \in \Omega$ . If  $\omega_1 \leq 0$ , then by Lemma 1, we have  $Q_1(\phi_\omega) > 0$  and  $\det[d''(\omega)] < 0$ . Thus, the matrix  $d''(\omega)$  has one positive eigenvalue and one negative eigenvalue. Therefore, by Theorem 2,  $T(\omega t)\phi_\omega$  is stable.

Next, we consider the case  $\omega_1 > 0$ . We put  $\xi = \sqrt{\gamma \left( \frac{4\omega_0}{\omega_1^2} - 1 \right)}$ . Then, by Lemma 1, we have

$$Q_1(\phi_\omega) = \frac{1}{\gamma} \sqrt{4\omega_0 - \omega_1^2} \{1 - g(\xi)\},$$

where  $g(\xi)$  is defined by (1.11) in Remark 1.

If  $g(\xi) < 1$ , then  $Q_1(\phi_\omega) > 0$  and  $\det[d''(\omega)] < 0$ . Thus,  $d''(\omega)$  has a positive eigenvalue, and by Theorem 2,  $T(\omega t)\phi_\omega$  is stable.

On the other hand, if  $g(\xi) > 1$ , then  $Q_1(\phi_\omega) < 0$  and  $\det[d''(\omega)] > 0$ . Moreover, since

$$\partial_{\omega_0}^2 d(\omega) = \partial_{\omega_0} Q_0(\phi_\omega) = \frac{-4\omega_1}{\sqrt{4\omega_0 - \omega_1^2} \{\gamma(4\omega_0 - \omega_1^2) + \omega_1^2\}} < 0,$$

we see that  $d''(\omega)$  is negative definite. Thus, it follows from Theorem 3 that  $T(\omega t)\phi_\omega$  is unstable.

Finally, by Remark 1, we see that  $g(\xi) < 1$  is equivalent to  $\omega_1 < 2\kappa\sqrt{\omega_0}$ , and that  $g(\xi) > 1$  is equivalent to  $\omega_1 > 2\kappa\sqrt{\omega_0}$ .  $\square$

The rest of the paper is organized as follows. In Section 2, we give a variational characterization of  $\phi_\omega$ . This part is essentially the same as Section 3 of [2], so we omit the details. In Section 3, we give the proof of Theorem 3. We divide the proof into two parts. In Subsection 3.1, we prove that if  $d''(\omega)$  is negative definite, then there exists an unstable direction  $\psi$ . In Subsection 3.2, we prove the instability of  $T(\omega t)\phi_\omega$  using the variational characterization of  $\phi_\omega$  and the unstable direction  $\psi$ .

## 2 Variational characterization

In this section, we give a variational characterization of  $\phi_\omega$ . Although  $\phi_\omega$  is given by (1.3) and (1.4) explicitly, we need such a variational characterization to prove stability and instability of solitary wave solutions  $T(\omega t)\phi_\omega$ .

Throughout this section, we assume that  $b > 0$ . The case  $b = 0$  is studied in Section 3 of [2], and the proof for the case  $b > 0$  is almost the same as that for  $b = 0$ , so we will omit the details.

For  $\omega \in \Omega$ , we define

$$\begin{aligned} L_\omega(v) &= \|\partial_x v\|_{L^2}^2 + \omega_0 \|v\|_{L^2}^2 + \omega_1 (i\partial_x v, v)_{L^2}, \\ S_\omega(v) &= \frac{1}{2}L_\omega(v) - \frac{1}{4}(i|v|^2\partial_x v, v)_{L^2} - \frac{b}{6}\|v\|_{L^6}^6, \\ K_\omega(v) &= L_\omega(v) - (i|v|^2\partial_x v, v)_{L^2} - b\|v\|_{L^6}^6, \end{aligned}$$

and consider the following minimization problem:

$$\mu(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega(v) = 0\}. \quad (2.1)$$

Note that (1.5) is equivalent to  $S'_\omega(\phi) = 0$  and that  $K_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1}$ .

We also define

$$\tilde{S}_\omega(v) = S_\omega(v) - \frac{1}{4}K_\omega(v) = \frac{1}{4}L_\omega(v) + \frac{b}{12}\|v\|_{L^6}^6.$$

**Lemma 2.** *Let  $\omega \in \Omega$ .*

- (1) *There exists a constant  $C_1 = C_1(\omega) > 0$  such that  $L_\omega(v) \geq C_1 \|v\|_{H^1}^2$  for all  $v \in H^1(\mathbb{R})$ .*
- (2)  *$\mu(\omega) > 0$ .*
- (3) *If  $v \in H^1(\mathbb{R})$  satisfies  $K_\omega(v) < 0$ , then  $\mu(\omega) < \tilde{S}_\omega(v)$ .*

*Proof.* (1) See Lemma 7 (1) of [2].

(2) Let  $v \in H^1(\mathbb{R}) \setminus \{0\}$  satisfy  $K_\omega(v) = 0$ . Then, by (1) and the Sobolev inequality, there exists  $C_2 > 0$  such that

$$\begin{aligned} C_1 \|v\|_{H^1}^2 &\leq L_\omega(v) = (i|v|^2 \partial_x v, v)_{L^2} + b \|v\|_{L^6}^6 \\ &\leq \|\partial_x v\|_{L^2} \|v\|_{L^6}^3 + b \|v\|_{L^6}^6 \leq \frac{C_1}{2} \|v\|_{H^1}^2 + C_2 \|v\|_{H^1}^6. \end{aligned}$$

Since  $v \neq 0$ , we have  $\|v\|_{H^1}^4 \geq \frac{C_1}{2C_2}$ . Thus, we have

$$\begin{aligned} \mu(\omega) &= \inf\{\tilde{S}_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega(v) = 0\} \\ &\geq \frac{1}{4} \inf\{L_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, K_\omega(v) = 0\} \geq \frac{C_1}{4} \sqrt{\frac{C_1}{2C_2}} > 0. \end{aligned}$$

(3) Let  $v \in H^1(\mathbb{R}) \setminus \{0\}$  satisfy  $K_\omega(v) < 0$ . Then, there exists  $\lambda_1 \in (0, 1)$  such that

$$K_\omega(\lambda_1 v) = \lambda_1^2 L_\omega(v) - \lambda_1^4 (i|v|^2 \partial_x v, v)_{L^2} - \lambda_1^6 b \|v\|_{L^6}^6 = 0.$$

Since  $v \neq 0$ , we have

$$\mu(\omega) \leq \tilde{S}_\omega(\lambda_1 v) = \frac{\lambda_1^2}{4} L_\omega(v) + \frac{\lambda_1^6 b}{12} \|v\|_{L^6}^6 < \tilde{S}_\omega(v).$$

This completes the proof. □

Let  $\mathcal{M}_\omega$  be the set of all minimizers for (2.1), i.e.,

$$\mathcal{M}_\omega = \{\varphi \in H^1(\mathbb{R}) \setminus \{0\} : S_\omega(\varphi) = \mu(\omega), K_\omega(\varphi) = 0\}.$$

Then, we obtain the following.

**Lemma 3.** For any  $\omega \in \Omega$ , we have  $\mathcal{M}_\omega = \{T(\theta)\phi_\omega : \theta \in \mathbb{R}^2\}$ . In particular, if  $v \in H^1(\mathbb{R})$  satisfies  $K_\omega(v) = 0$  and  $v \neq 0$ , then  $S_\omega(\phi_\omega) \leq S_\omega(v)$ .

The proof of Lemma 3 is almost the same as that of Lemma 10 of [2], so we omit it.

The following lemma plays an important role in the proof of Lemma 12.

**Lemma 4.** If  $v \in H^1(\mathbb{R})$  satisfies  $\langle K'_\omega(\phi_\omega), v \rangle = 0$ , then  $\langle S''_\omega(\phi_\omega)v, v \rangle \geq 0$ .

*Proof.* Let  $v \in H^1(\mathbb{R})$  satisfy  $\langle K'_\omega(\phi_\omega), v \rangle = 0$ . Since  $K_\omega(\phi_\omega) = 0$  and  $\langle K'_\omega(\phi_\omega), \phi_\omega \rangle \neq 0$ , by the implicit function theorem, there exist a constant  $\delta > 0$  and a  $C^2$ -function  $\gamma : (-\delta, \delta) \rightarrow \mathbb{R}$  such that  $\gamma(0) = 0$  and

$$K_\omega(\phi_\omega + sv + \gamma(s)\phi_\omega) = 0, \quad s \in (-\delta, \delta). \quad (2.2)$$

Taking  $\delta$  smaller if necessary, we also have  $\phi_\omega + sv + \gamma(s)\phi_\omega \neq 0$  for  $s \in (-\delta, \delta)$ .

Differentiating (2.2) at  $s = 0$ , we have

$$0 = \langle K'_\omega(\phi_\omega), v \rangle + \gamma'(0) \langle K'_\omega(\phi_\omega), \phi_\omega \rangle.$$

Since  $\langle K'_\omega(\phi_\omega), v \rangle = 0$  and  $\langle K'_\omega(\phi_\omega), \phi_\omega \rangle \neq 0$ , we have  $\gamma'(0) = 0$ .

Moreover, since  $\phi_\omega \in \mathcal{M}_\omega$  by Lemma 3, it follows from (2.2) that the function  $s \mapsto S_\omega(\phi_\omega + sv + \gamma(s)\phi_\omega)$  has a local minimum at  $s = 0$ . Thus, we have

$$\begin{aligned} 0 &\leq \frac{d^2}{ds^2} S_\omega(\phi_\omega + sv + \gamma(s)\phi_\omega) \Big|_{s=0} \\ &= \langle S''_\omega(\phi_\omega)(v + \gamma'(0)\phi_\omega), v + \gamma'(0)\phi_\omega \rangle + \langle S'_\omega(\phi_\omega), \gamma''(0)\phi_\omega \rangle \\ &= \langle S''_\omega(\phi_\omega)v, v \rangle. \end{aligned}$$

This completes the proof.  $\square$

### 3 Proof of Theorem 3

In this section, we give the proof of Theorem 3. We divide the proof into two parts. In Subsection 3.1, we prove that if  $d''(\omega)$  is negative definite, then there exists an unstable direction  $\psi$  (see Lemma 6). In Subsection 3.2, we prove the instability of  $T(\omega t)\phi_\omega$  using the variational characterization of  $\phi_\omega$  and the unstable direction  $\psi$  (see Proposition 1). Theorem 3 follows from Lemma 6 and Proposition 1.



### 3.1 Existence of unstable direction

**Lemma 5.**  $\langle S''_\omega(\phi_\omega)\phi_\omega, \phi_\omega \rangle < 0$ .

*Proof.* Since the function

$$(0, \infty) \ni \lambda \mapsto S_\omega(\lambda\phi_\omega) = \frac{\lambda^2}{2}L_\omega(\phi_\omega) - \frac{\lambda^4}{4}(i|\phi_\omega|^2\partial_x\phi_\omega, \phi_\omega)_{L^2} - \frac{\lambda^6 b}{6}\|\phi_\omega\|_{L^6}^6$$

has a strictly local maximum at  $\lambda = 1$ , we have

$$0 > \frac{d^2}{d\lambda^2}S_\omega(\lambda\phi_\omega)\big|_{\lambda=1} = \langle S''_\omega(\phi_\omega)\phi_\omega, \phi_\omega \rangle.$$

This completes the proof.  $\square$

**Lemma 6.** *Assume that  $d''(\hat{\omega})$  is negative definite. Then there exists  $\psi \in H^1(\mathbb{R})$  such that*

$$\langle Q'_0(\phi_{\hat{\omega}}), \psi \rangle = \langle Q'_1(\phi_{\hat{\omega}}), \psi \rangle = 0, \quad \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\psi, \psi \rangle < 0.$$

*Proof.* For  $(s, \omega)$  near  $(0, \hat{\omega})$  in  $\mathbb{R} \times \Omega$ , we define

$$F(s, \omega) := \begin{bmatrix} Q_0(s\phi_{\hat{\omega}} + \phi_\omega) - Q_0(\phi_{\hat{\omega}}) \\ Q_1(s\phi_{\hat{\omega}} + \phi_\omega) - Q_1(\phi_{\hat{\omega}}) \end{bmatrix}.$$

Then, we have  $F(0, \hat{\omega}) = 0$ . Moreover, since  $D_\omega F(0, \hat{\omega}) = d''(\hat{\omega})$  is negative definite and invertible, by the implicit function theorem, there exist a constant  $\delta > 0$  and a  $C^1$ -function  $\gamma : (-\delta, \delta) \rightarrow \Omega$  such that  $\gamma(0) = \hat{\omega}$  and

$$Q_0(s\phi_{\hat{\omega}} + \phi_{\gamma(s)}) = Q_0(\phi_{\hat{\omega}}), \quad Q_1(s\phi_{\hat{\omega}} + \phi_{\gamma(s)}) = Q_1(\phi_{\hat{\omega}})$$

for  $s \in (-\delta, \delta)$ . We define  $\varphi_s := s\phi_{\hat{\omega}} + \phi_{\gamma(s)}$  for  $s \in (-\delta, \delta)$ , and

$$w_j := \partial_{\omega_j}\phi_\omega|_{\omega=\hat{\omega}} \quad (j = 0, 1), \quad \psi := \partial_s\varphi_s|_{s=0} = \phi_{\hat{\omega}} + \sum_{j=0}^1 \gamma'_j(0)w_j.$$

Then, for  $j = 0, 1$ , we have

$$\begin{aligned} 0 &= \frac{d}{ds}Q_j(\varphi_s)|_{s=0} = \langle Q'_j(\phi_{\hat{\omega}}), \psi \rangle \\ &= \langle Q'_j(\phi_{\hat{\omega}}), \phi_{\hat{\omega}} \rangle + \sum_{k=0}^1 \gamma'_k(0) \langle Q'_j(\phi_{\hat{\omega}}), w_k \rangle. \end{aligned} \tag{3.1}$$

Moreover, differentiating

$$0 = S'_\omega(\phi_\omega) = E'(\phi_\omega) + \sum_{k=0}^1 \omega_k Q'_k(\phi_\omega),$$

with respect to  $\omega_j$  for  $j = 0, 1$ , we have

$$\begin{aligned} 0 &= E''(\phi_\omega)(\partial_{\omega_j} \phi_\omega) + \sum_{k=0}^1 \omega_k Q''_k(\phi_\omega)(\partial_{\omega_j} \phi_\omega) + Q'_j(\phi_\omega) \\ &= S''_\omega(\phi_\omega)(\partial_{\omega_j} \phi_\omega) + Q'_j(\phi_\omega). \end{aligned} \quad (3.2)$$

By (3.1) and (3.2), we have

$$\begin{aligned} \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\psi, \psi \rangle &= \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\phi_{\hat{\omega}}, \phi_{\hat{\omega}} \rangle + 2 \sum_{j=0}^1 \gamma'_j(0) \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})w_j, \phi_{\hat{\omega}} \rangle \\ &\quad + \sum_{j,k=0}^1 \gamma'_j(0) \gamma'_k(0) \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})w_j, w_k \rangle \\ &= \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\phi_{\hat{\omega}}, \phi_{\hat{\omega}} \rangle - 2 \sum_{j=0}^1 \gamma'_j(0) \langle Q'_j(\phi_{\hat{\omega}}), \phi_{\hat{\omega}} \rangle - \sum_{j,k=0}^1 \gamma'_j(0) \gamma'_k(0) \langle Q'_j(\phi_{\hat{\omega}}), w_k \rangle \\ &= \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\phi_{\hat{\omega}}, \phi_{\hat{\omega}} \rangle + \sum_{j,k=0}^1 \gamma'_j(0) \gamma'_k(0) \langle Q'_j(\phi_{\hat{\omega}}), w_k \rangle \\ &= \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\phi_{\hat{\omega}}, \phi_{\hat{\omega}} \rangle + \sum_{j,k=0}^1 \gamma'_j(0) \gamma'_k(0) \partial_{\omega_j} \partial_{\omega_k} d(\hat{\omega}). \end{aligned}$$

Since  $d''(\hat{\omega})$  is negative definite, it follows from Lemma 5 that

$$\langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\psi, \psi \rangle \leq \langle S''_{\hat{\omega}}(\phi_{\hat{\omega}})\phi_{\hat{\omega}}, \phi_{\hat{\omega}} \rangle < 0.$$

This completes the proof.  $\square$

### 3.2 Proof of instability

In this subsection, we prove the following.

**Proposition 1.** *Let  $\omega \in \Omega$ , and assume that there exists  $\psi \in H^1(\mathbb{R})$  such that*

$$\langle Q'_0(\phi_\omega), \psi \rangle = \langle Q'_1(\phi_\omega), \psi \rangle = 0, \quad \langle S''_\omega(\phi_\omega)\psi, \psi \rangle < 0. \quad (3.3)$$

*Then, the solitary wave solution  $T(\omega t)\phi_\omega$  of (1.1) is unstable.*

To prove Proposition 1, we use the argument of Gonçalves Ribeiro [3] (see also [17, 4]) with some modifications. Throughout this subsection, we fix  $\omega \in \Omega$ , and assume that  $\psi \in H^1(\mathbb{R})$  satisfies (3.3).

**Lemma 7.** *There exists a constant  $\lambda_0 > 0$  such that*

$$S_\omega(\phi_\omega + \lambda\psi) < S_\omega(\phi_\omega)$$

for all  $\lambda \in (-\lambda_0, 0) \cup (0, \lambda_0)$ .

*Proof.* By Taylor's expansion, for  $\lambda \in \mathbb{R}$ , we have

$$\begin{aligned} S_\omega(\phi_\omega + \lambda\psi) &= S_\omega(\phi_\omega) + \lambda \langle S'_\omega(\phi_\omega), \psi \rangle + \lambda^2 \int_0^1 (1-s) \langle S''_\omega(\phi_\omega + s\lambda\psi)\psi, \psi \rangle ds \\ &= S_\omega(\phi_\omega) + \lambda^2 \int_0^1 (1-s) \langle S''_\omega(\phi_\omega + s\lambda\psi)\psi, \psi \rangle ds. \end{aligned}$$

Since  $\langle S''_\omega(\phi_\omega)\psi, \psi \rangle < 0$ , by the continuity of  $\lambda \mapsto \langle S''_\omega(\phi_\omega + \lambda\psi)\psi, \psi \rangle$ , there exists  $\lambda_0 > 0$  such that

$$\langle S''_\omega(\phi_\omega + \lambda\psi)\psi, \psi \rangle \leq \frac{1}{2} \langle S''_\omega(\phi_\omega)\psi, \psi \rangle$$

for all  $\lambda \in (-\lambda_0, \lambda_0)$ . Thus, for  $\lambda \in (-\lambda_0, 0) \cup (0, \lambda_0)$ , we have

$$S_\omega(\phi_\omega + \lambda\psi) \leq S_\omega(\phi_\omega) + \frac{\lambda^2}{4} \langle S''_\omega(\phi_\omega)\psi, \psi \rangle < S_\omega(\phi_\omega).$$

This completes the proof. □

For  $u \in H^1(\mathbb{R})$ , we define

$$T'_0 u = iu, \quad T'_1 u = -\partial_x u.$$

Then, by (1.8) and (1.10), we have

$$\partial_{\theta_j} T(\theta)u = T(\theta)T'_j u = T'_j T(\theta)u, \quad \langle Q'_j(u), v \rangle = (T'_j u, iv)_{L^2} \quad (3.4)$$

for  $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$ ,  $u, v \in H^1(\mathbb{R})$  and  $j = 0, 1$ . We denote  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ .

**Lemma 8.** *There exist a constant  $\varepsilon_0 > 0$  and a  $C^1$ -function*

$$\alpha = (\alpha_0, \alpha_1) : U_{\varepsilon_0}(\phi_\omega) \rightarrow \mathbb{T} \times \mathbb{R}$$

*such that  $\alpha(\phi_\omega) = 0$ , and*

- (1)  $\alpha(T(\xi)u) = \alpha(u) + \xi$  for all  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $\xi \in \mathbb{T} \times \mathbb{R}$ .
- (2)  $(T'_j u, T(\alpha(u))\phi_\omega)_{L^2} = 0$  for all  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $j = 0, 1$ .
- (3) *There exists  $\rho > 0$  such that*

$$\sum_{j,k=0}^1 (T'_j u, T(\alpha(u))T'_k \phi_\omega)_{L^2} \zeta_j \zeta_k \geq \rho |\zeta|^2$$

*for all  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$ .*

*Proof.* See Section 3 of [3]. □

For  $u \in U_{\varepsilon_0}(\phi_\omega)$ , we define

$$H(u) = [h_{jk}(u)]_{j,k=0,1}, \quad h_{jk}(u) = (T'_j u, T(\alpha(u))T'_k \phi_\omega)_{L^2}.$$

Then, by Lemma 8 (1), we have

$$h_{jk}(T(\xi)u) = (T(\xi)T'_j u, T(\alpha(u) + \xi)T'_k \phi_\omega)_{L^2} = h_{jk}(u) \quad (3.5)$$

for  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $\xi \in \mathbb{T} \times \mathbb{R}$ .

Moreover, differentiating Lemma 8 (2) with respect to  $u$ , we have

$$\sum_{k=0}^1 h_{jk}(u) \langle \alpha'_k(u), w \rangle = (T(\alpha(u))T'_j \phi_\omega, w)_{L^2} \quad (3.6)$$

for  $u \in U_{\varepsilon_0}(\phi_\omega)$ ,  $w \in H^1(\mathbb{R})$  and  $j = 0, 1$ . By Lemma 8 (3), the matrix  $H(u)$  is invertible, and we denote the inverse  $H(u)^{-1}$  by  $G(u) = [g_{jk}(u)]$ . Then, there exists a constant  $C > 0$  such that

$$|g_{jk}(u)| \leq C \text{ for all } u \in U_{\varepsilon_0}(\phi_\omega), \quad j, k = 0, 1. \quad (3.7)$$

For  $j = 0, 1$  and  $u \in U_{\varepsilon_0}(\phi_\omega)$ , we define

$$a_j(u) := \sum_{k=0}^1 g_{jk}(u) T(\alpha(u))T'_k \phi_\omega.$$

Since  $\phi_\omega \in H^2(\mathbb{R})$ , we see that  $a_j(u) \in H^1(\mathbb{R})$ , it follows from (3.6) that

$$\langle \alpha'_j(u), w \rangle = (a_j(u), w)_{L^2}, \quad w \in H^1(\mathbb{R}).$$

By (3.5) and Lemma 8 (1), for  $j = 0, 1$ , we have

$$a_j(T(\xi)u) = T(\xi)a_j(u) \text{ for all } u \in U_{\varepsilon_0}(\phi_\omega), \xi \in \mathbb{T} \times \mathbb{R}. \quad (3.8)$$

Moreover, by (3.7), there exists a constant  $C > 0$  such that

$$\|a_j(u)\|_{H^1} \leq C \text{ for all } u \in U_{\varepsilon_0}(\phi_\omega), j = 0, 1. \quad (3.9)$$

Next, for  $u \in U_{\varepsilon_0}(\phi_\omega)$ , we define

$$A(u) = (iu, T(\alpha(u))\psi)_{L^2}, \quad (3.10)$$

$$q(u) = T(\alpha(u))\psi + \sum_{j=0}^1 (iu, T(\alpha(u))T'_j\psi)_{L^2} ia_j(u). \quad (3.11)$$

Then, since  $\psi, a_0(u), a_1(u) \in H^1(\mathbb{R})$ , we see that  $q(u) \in H^1(\mathbb{R})$ .

**Lemma 9.** For  $u \in U_{\varepsilon_0}(\phi_\omega)$ ,

$$(1) \quad A(T(\xi)u) = A(u), \quad q(T(\xi)u) = T(\xi)q(u) \text{ for all } \xi \in \mathbb{T} \times \mathbb{R}.$$

$$(2) \quad \langle A'(u), w \rangle = (q(u), iw)_{L^2} \text{ for } w \in H^1(\mathbb{R}).$$

$$(3) \quad q(\phi_\omega) = \psi.$$

$$(4) \quad \langle Q'_j(u), q(u) \rangle = 0 \text{ for } j = 0, 1.$$

*Proof.* (1) By Lemma 8 (1), we have

$$\begin{aligned} A(T(\xi)u) &= (iT(\xi)u, T(\alpha(u) + \xi)\psi)_{L^2} \\ &= (iT(\xi)u, T(\xi)T(\alpha(u))\psi)_{L^2} = A(u). \end{aligned}$$

Moreover, by (3.8), we have

$$\begin{aligned} q(T(\xi)u) &= T(\xi)T(\alpha(u))\psi + \sum_{j=0}^1 (iT(\xi)u, T(\xi)T(\alpha(u))T'_j\psi)_{L^2} ia_j(T(\xi)u) \\ &= T(\xi)q(u). \end{aligned}$$

(2) For  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $w \in H^1(\mathbb{R})$ , we have

$$\begin{aligned}\langle A'(u), w \rangle &= (iw, T(\alpha(u))\psi)_{L^2} + \sum_{j=0}^1 \langle \alpha'_j(u), w \rangle (iu, T(\alpha(u))T'_j\psi)_{L^2} \\ &= (iw, T(\alpha(u))\psi)_{L^2} + \sum_{j=0}^1 (iu, T(\alpha(u))T'_j\psi)_{L^2} (a_j(u), w)_{L^2} \\ &= (q(u), iw)_{L^2}.\end{aligned}$$

(3) By (3.4) and the assumption (3.3), we have

$$(i\phi_\omega, T'_j\psi)_{L^2} = (T'_j\phi_\omega, i\psi)_{L^2} = \langle Q'_j(\phi_\omega), \psi \rangle = 0.$$

Moreover, since  $\alpha(\phi_\omega) = 0$ , by (3.11), we have  $q(\phi_\omega) = \psi$ .

(4) For  $u \in H^2(\mathbb{R}) \cap U_{\varepsilon_0}(\phi_\omega)$ , by (1) and (2), we have

$$0 = \partial_{\xi_j} A(T(\xi)u) \Big|_{\xi=0} = \langle A'(u), T'_j u \rangle = (q(u), iT'_j u)_{L^2}.$$

By density argument, we have  $(q(u), iT'_j u)_{L^2} = 0$  for all  $u \in U_{\varepsilon_0}(\phi_\omega)$ .

Thus, we have  $\langle Q'_j(u), q(u) \rangle = (T'_j u, iq(u))_{L^2} = 0$  for  $u \in U_{\varepsilon_0}(\phi_\omega)$ .  $\square$

For  $u \in U_{\varepsilon_0}(\phi_\omega)$ , we define

$$P(u) := \langle E'(u), q(u) \rangle.$$

We remark that by Lemma 9 (4), we have

$$P(u) = \langle S'_\omega(u), q(u) \rangle, \quad u \in U_{\varepsilon_0}(\phi_\omega). \quad (3.12)$$

**Lemma 10.** *Let  $I$  be an interval of  $\mathbb{R}$ . Let  $u \in C(I, H^1(\mathbb{R})) \cap C^1(I, H^{-1}(\mathbb{R}))$  be a solution of (1.1), and assume that  $u(t) \in U_{\varepsilon_0}(\phi_\omega)$  for all  $t \in I$ . Then,*

$$\frac{d}{dt} A(u(t)) = P(u(t))$$

for all  $t \in I$ .

*Proof.* By Lemma 4.6 of [4] and Lemma 9 (2), we see that  $t \mapsto A(u(t))$  is a  $C^1$ -function on  $I$ , and

$$\frac{d}{dt} A(u(t)) = \langle i\partial_t u(t), q(u(t)) \rangle$$

for all  $t \in I$ . Since  $u(t)$  is a solution of (1.1), we have

$$\langle i\partial_t u(t), q(u(t)) \rangle = \langle E'(u(t)), q(u(t)) \rangle = P(u(t))$$

for all  $t \in I$ . This completes the proof.  $\square$

**Lemma 11.** *There exist constants  $\lambda_1 > 0$  and  $\varepsilon_1 \in (0, \varepsilon_0)$  such that*

$$S_\omega(u + \lambda q(u)) \leq S_\omega(u) + \lambda P(u)$$

for all  $\lambda \in (-\lambda_1, \lambda_1)$  and  $u \in U_{\varepsilon_1}(\phi_\omega)$ .

*Proof.* For  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $\lambda \in \mathbb{R}$ , by Taylor's expansion, we have

$$S_\omega(u + \lambda q(u)) = S_\omega(u) + \lambda P(u) + \lambda^2 \int_0^1 (1-s) R(\lambda s, u) ds, \quad (3.13)$$

where we used (3.12) and put

$$R(\lambda, u) := \langle S''_\omega(u + \lambda q(u)) q(u), q(u) \rangle.$$

Here, we remark that

$$\begin{aligned} P(T(\xi)u) &= \langle S'_\omega(T(\xi)u), T(\xi)q(u) \rangle = P(u), \\ R(\lambda, T(\xi)u) &= \langle S''_\omega(T(\xi)(u + \lambda q(u))) T(\xi)q(u), T(\xi)q(u) \rangle = R(\lambda, u) \end{aligned}$$

for  $\xi \in \mathbb{T} \times \mathbb{R}$ ,  $\lambda \in \mathbb{R}$  and  $u \in H^1(\mathbb{R})$ . Moreover, since

$$R(0, \phi_\omega) = \langle S''_\omega(\phi_\omega) q(\phi_\omega), q(\phi_\omega) \rangle = \langle S''_\omega(\phi_\omega) \psi, \psi \rangle < 0,$$

by the continuity of  $R(\lambda, u)$  with respect to  $\lambda$  and  $u$ , there exist constants  $\lambda_1 > 0$  and  $\varepsilon_1 \in (0, \varepsilon_0)$  such that  $R(\lambda, u) < 0$  for all  $\lambda \in (-\lambda_1, \lambda_1)$  and  $u \in U_{\varepsilon_1}(\phi_\omega)$ . Thus, by (3.13), we have

$$S_\omega(u + \lambda q(u)) \leq S_\omega(u) + \lambda P(u)$$

for all  $\lambda \in (-\lambda_1, \lambda_1)$  and  $u \in U_{\varepsilon_1}(\phi_\omega)$ .  $\square$

**Lemma 12.** *There exist constants  $\varepsilon_2 \in (0, \varepsilon_1)$  and  $\lambda_2 \in (0, \lambda_1)$  that satisfy the following. For any  $u \in U_{\varepsilon_2}(\phi_\omega)$ , there exists  $\Lambda(u) \in (-\lambda_2, \lambda_2)$  such that*

$$K_\omega(u + \Lambda(u)q(u)) = 0, \quad u + \Lambda(u)q(u) \neq 0.$$

*Proof.* First, since  $\langle S''_\omega(\phi_\omega)\psi, \psi \rangle < 0$ , by Lemma 4, we have  $\langle K'_\omega(\phi_\omega), \psi \rangle \neq 0$ . Thus, without loss of generality, we may assume that  $\langle K'_\omega(\phi_\omega), \psi \rangle > 0$ .

For  $u \in U_{\varepsilon_0}(\phi_\omega)$  and  $\lambda \in \mathbb{R}$ , we have

$$K_\omega(u + \lambda q(u)) = K_\omega(u) + \lambda \int_0^1 \langle K'_\omega(u + s\lambda q(u)), q(u) \rangle ds. \quad (3.14)$$

Since  $\langle K'_\omega(\phi_\omega), q(\phi_\omega) \rangle = \langle K'_\omega(\phi_\omega), \psi \rangle > 0$ , by the continuity of the function  $\langle K'_\omega(u + \lambda q(u)), q(u) \rangle$  with respect to  $\lambda$  and  $u$ , there exist constants  $\lambda_2 \in (0, \lambda_1)$  and  $\varepsilon_2 \in (0, \varepsilon_1)$  such that

$$\langle K'_\omega(u + \lambda q(u)), q(u) \rangle \geq \frac{1}{2} \langle K'_\omega(\phi_\omega), \psi \rangle \quad (3.15)$$

for all  $\lambda \in [-\lambda_2, \lambda_2]$  and  $u \in U_{\varepsilon_2}(\phi_\omega)$ . Moreover, since  $K_\omega(\phi_\omega) = 0$ , taking  $\varepsilon_2$  smaller if necessary, we have

$$|K_\omega(u)| < \frac{\lambda_2}{2} \langle K'_\omega(\phi_\omega), \psi \rangle, \quad u \in U_{\varepsilon_2}(\phi_\omega). \quad (3.16)$$

Let  $u \in U_{\varepsilon_2}(\phi_\omega)$ . If  $K_\omega(u) < 0$ , then it follows from (3.14)–(3.16) that

$$\begin{aligned} K_\omega(u + \lambda_2 q(u)) &= K_\omega(u) + \lambda_2 \int_0^1 \langle K'_\omega(u + s\lambda_2 q(u)), q(u) \rangle ds \\ &> -\frac{\lambda_2}{2} \langle K'_\omega(\phi_\omega), \psi \rangle + \frac{\lambda_2}{2} \langle K'_\omega(\phi_\omega), \psi \rangle = 0. \end{aligned}$$

Since the function  $\lambda \mapsto K_\omega(u + \lambda q(u))$  is continuous, there exists  $\Lambda(u) \in (0, \lambda_2)$  such that

$$K_\omega(u + \Lambda(u)q(u)) = 0. \quad (3.17)$$

Similarly, if  $K_\omega(u) > 0$ , then we have

$$\begin{aligned} K_\omega(u - \lambda_2 q(u)) &= K_\omega(u) - \lambda_2 \int_0^1 \langle K'_\omega(u - s\lambda_2 q(u)), q(u) \rangle ds \\ &< \frac{\lambda_2}{2} \langle K'_\omega(\phi_\omega), \psi \rangle - \frac{\lambda_2}{2} \langle K'_\omega(\phi_\omega), \psi \rangle = 0. \end{aligned}$$

Thus, there exists  $\Lambda(u) \in (-\lambda_2, 0)$  such that (3.17). If  $K_\omega(u) = 0$ , taking  $\Lambda(u) = 0$ , (3.17) is satisfied.

Finally, by (3.9) and (3.11), taking  $\lambda_2$  and  $\varepsilon_2$  smaller if necessary, we have  $u + \Lambda(u)q(u) \neq 0$  for all  $u \in U_{\varepsilon_2}(\phi_\omega)$ . This completes the proof.  $\square$



**Lemma 13.** *Let  $\lambda_2$  and  $\varepsilon_2$  be the positive constants given in Lemma 12. Then,*

$$S_\omega(\phi_\omega) \leq S_\omega(u) + \lambda_2|P(u)|$$

for all  $u \in U_{\varepsilon_2}(\phi_\omega)$ .

*Proof.* By Lemma 12, for any  $u \in U_{\varepsilon_2}(\phi_\omega)$ , there exists  $\Lambda(u) \in (-\lambda_2, \lambda_2)$  such that  $K_\omega(u + \Lambda(u)q(u)) = 0$  and  $u + \Lambda(u)q(u) \neq 0$ . Then, it follows from Lemma 3 that

$$S_\omega(\phi_\omega) \leq S_\omega(u + \Lambda(u)q(u)), \quad u \in U_{\varepsilon_2}(\phi_\omega). \quad (3.18)$$

Thus, by Lemma 11 and (3.18), for  $u \in U_{\varepsilon_2}(\phi_\omega)$ , we have

$$\begin{aligned} S_\omega(\phi_\omega) &\leq S_\omega(u + \Lambda(u)q(u)) \leq S_\omega(u) + \Lambda(u)P(u) \\ &\leq S_\omega(u) + |\Lambda(u)||P(u)| \leq S_\omega(u) + \lambda_2|P(u)|. \end{aligned}$$

This completes the proof.  $\square$

We are now in a position to give the Proof of Proposition 1.

*Proof of Proposition 1.* Suppose that  $T(\omega t)\phi_\omega$  is stable. For  $\lambda$  close to 0, let  $u_\lambda(t)$  be the solution of (1.1) with  $u_\lambda(0) = \phi_\omega + \lambda\psi$ . Since  $T(\omega t)\phi_\omega$  is stable, there exists  $\lambda_3 \in (0, \lambda_0)$  such that if  $|\lambda| < \lambda_3$ , then  $u_\lambda(t) \in U_{\varepsilon_2}(\phi_\omega)$  for all  $t \geq 0$ . Moreover, by the definition (3.10) of  $A$ , there exists  $C_1 > 0$  such that  $|A(v)| \leq C_1$  for all  $v \in U_{\varepsilon_2}(\phi_\omega)$ .

Let  $\lambda \in (-\lambda_3, 0) \cup (0, \lambda_3)$ . Then, by Lemma 7, we have

$$\delta_\lambda := S_\omega(\phi_\omega) - S_\omega(u_\lambda(0)) > 0.$$

Moreover, by Lemma 13 and the conservation of  $S_\omega$ , we have

$$0 < \delta_\lambda = S_\omega(\phi_\omega) - S_\omega(u_\lambda(t)) \leq \lambda_2|P(u_\lambda(t))|, \quad t \geq 0.$$

Since  $t \mapsto P(u_\lambda(t))$  is continuous, we see that either (i)  $P(u_\lambda(t)) \geq \delta_\lambda/\lambda_2$  for all  $t \geq 0$ , or (ii)  $P(u_\lambda(t)) \leq -\delta_\lambda/\lambda_2$  for all  $t \geq 0$ . Moreover, by Lemma 10, we have

$$\frac{d}{dt}A(u_\lambda(t)) = P(u_\lambda(t)), \quad t \geq 0.$$

Therefore, we see that  $A(u_\lambda(t)) \rightarrow \infty$  as  $t \rightarrow \infty$  for the case (i), and  $A(u_\lambda(t)) \rightarrow -\infty$  as  $t \rightarrow \infty$  for the case (ii). This contradicts the fact that  $|A(u_\lambda(t))| \leq C_1$  for all  $t \geq 0$ . Hence,  $T(\omega t)\phi_\omega$  is unstable.  $\square$

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